Math 206B Lecture 7 Notes

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1 The Robinson-Schensted Algorithm

1.1 Description of the algorithm

Recall Burnside's identity:

Proposition 1.1.

$$\sum_{\lambda} (f^{\lambda})^2 = n!,$$

where

$$f^{\lambda} = \dim(S^{\lambda}) = \chi^{\lambda}(1) = \#\,\mathrm{SYT}(\lambda).$$

We want a bijection between the symmetric group and the set of pairs (A, B), where A, B are $\mathrm{SYT}(\lambda)$ for some partition λ of n. We usually write the bijection Φ as $\sigma \mapsto (P, Q)$, where P is called the **insertion tableau** and Q is called the **recording tableau**.

Example 1.1. Let $\sigma = 4\ 2\ 7\ 3\ 6\ 1\ 5$ be a permutation. For each partial reading of the string representing σ, the algorithm will produce an outcome.

First, we start with the 4. Here are our two tableaux.

$$\begin{array}{|c|}\hline 4 \\ \hline \end{array} \qquad\qquad \begin{array}{|c|}\hline 1 \\ \hline \end{array}$$

Now let's add the 2. It can't go to the right of the 4, so it pushes the 4 down. We record the move into our right (recording) tableau:

$$\begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline \end{array} \qquad\qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$$

Now we add the 7. It is bigger than the 2, so it goes to the right.

$$\begin{array}{|c|c|}\hline 2 & 7 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad\qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Now we add the 3. It bumps down the smallest number in the first row that is bigger than it, the 7. When the 7 moves down, it does not need to bump the 4:

$$\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & 7 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

Now let's add in the 6.

$$\begin{array}{|c|c|c|}\hline 2 & 3 & 6 \\ \hline 4 & 7 \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}$$

What happens when we put in the 1? It is smaller than 2, so it bumps the 2 down. Doing the same process, the 2 is going to bump down the 4:

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 7 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}$$

Finally, we can add in the 5:

$$P = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 4 & 7 \\ \cline{1-2} \end{array} \qquad Q = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} \end{array}$$

Theorem 1.1 (R-S, 1961)**.** *The map* $\Phi : S_n \to \{(A, B) : A, B \in \mathrm{SYT}(\lambda), |\lambda| = n\}$ *is a bijection.*

Proof. Here are the steps:

1. The map Φ is well defined.
2. $P, Q \in \mathrm{SYT}(\lambda)$ for some partition λ of n.
3. Φ^{-1} is well-defined.

The first two are easy to convince yourself of. For the third, run the algorithm in reverse and see that it outputs the original permutation.[1] $\square$

[1] Back in the 60s, people used to listen to songs backwards to find hidden messages. This is same same, except there is actually a message if you go backwards.

1.2 Properties of the RS algorithm

This bijection is natural in some sense. It actually exhibits the following remarkable property!

Theorem 1.2. *Suppose* $\Phi(\sigma) = (P, Q)$*. Then* $\Phi(\sigma^{-1}) = (Q, P)$*.*

We will not prove this now, but we will prove it later in the course.

Theorem 1.3 (Schensted)**.** *Let* $\Phi(\sigma) = (P, Q)$*, where* $P, Q \in \mathrm{SYT}(\lambda)$*. Then* λ_1 *is the length of the longest increasing subsequence in* σ*.*

Proof. Proceed by induction. We claim that λ_1 of P_i is the longest increasing subsequence of $\sigma = \sigma_1 \ \cdots \ \sigma_i$, where P_i is the insertion tableau at the i-th step. Suppose we have $[a_1 \ \cdots \ a_r]\sigma_{i+1}$, where $a_1 \cdots a_r$ is the longest increasing subsequence in the permutation. If $\sigma_{i+1} > a_r$, then we must add a number to the first row because nothing gets bumped down. If $a_r > \sigma_{i+1}$, then something in the first row gets bumped down. $\square$

How does one come up with an algorithm like this? Schensted was a graduate student at either Berkeley or Stanford.[2] Schensted had a roommate, Floyd, who later became a famous computer scientist. They were interested in sorting things like solitaire (place the newest card on the heap with the smallest number). Schensted saw the above property and found it interesting. The rest of the story will have to wait until next time.

[2] Professor Pak doesn't remember which.